Statistical closure and the logistic map

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We consider the chaotic logistic map with stochastic initial conditions. All initial conditions assume a Gaussian distribution centered in the unit interval with a small dispersion. We show that the system exhibits behavior characterized by three different regimes (called initial, transient, and final). The initial interval is characterized by the Gaussian closure being accurate and the evolution of the system dominated by the evolution of the mean. The transient interval is characterized by rapid growth of all cumulants and a breakdown of Gaussian closure. We identify this period as the Suzuki scaling regime. An alternative closure scheme based on the beta distribution is also introduced. We find that the evolution equations for the mean and dispersion based on a beta distribution closure give accurate predictions over all iterations. This type of closure assumes nothing about the vanishing of higher-order cumulants (in fact, cumulants of all order are nonvanishing). The possible relevance of these results to clump kinetics is also addressed.

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The statistical closure problem arises whenever one is attempting to solve a nonlinear stochastic equation. The wide use of closure in statistical models raises the issue of their validity as a system evolves. That is, given a closure relation at some initial time, does the closure relation remain valid for all time? It is of course only possible to answer this question definitively for models which can be solved independent of the closure approximation (either numerically or, if one is lucky, analytically). Only for this class of problems can a direct comparison be made between the true probability distribution and a distribution based on an expansion in moments derived from the closure evolution equations. Nonlinear maps with either external noise [1] or stochastic initial conditions offer an excellent arena for testing the viability of closure relations. They exhibit complicated dynamics while at the same time being amenable to numerical calculation. In particular, the logistic map possesses much of the structure of more complicated maps including fixed points, period doubling, and chaos. In this paper we will concern ourselves with the statistical closure problem as it relates to the logistic map with stochastic initial conditions:

$$x_{n+1}=\mu x_n(1-x_n) . \tag{1}$$

We will be concerned primarily with the case $\mu=4$. For simplicity we will refer to this as the chaotic logistic map. The specific problem we wish to address is, given a Gaussian distribution of initial values centered in the unit interval, how does this probability distribution evolve when each point of the function evolves according to the logistic map? In particular, how well do the evolution equations for the moments (which are based on a given closure relation) do in predicting the correct dynamics? In what regimes and under what circumstances do the closure relations work? Our choice of Gaussian initial conditions is purely arbitrary and done for ease of analysis (the evolution equations for the moments are easy to derive for a Gaussian closure) and besides, there is no *a priori* reason for choosing any other set of stochastic initial conditions. The questions which we address in this paper constitute an admittedly restricted set of parameter space, but we feel that something can be learned by studying the closure issue in the context of a well understood dynamical process. The logistic map provides an elementary environment in which to study not only the closure problem but also the problem of how a compact group of particles evolve when under some type of chaotic influence. For example, Sommerer and Ott [2] have looked at the behavior of an aggregate of tracer particles on the surface of an unstable fluid. The theoretical basis of their work relied on characterizing the evolution of a particle at a specific time via a random map. Thus, the above problem presents an effective toy model for understanding clump kinetics in turbulent fluids and plasmas.

Gaussian closure applied to the logistic map for arbitrary μ implies a two-parameter description of the dynamics given by

$$M_{n+1}=\mu M_n(1-M_n)-\mu\sigma_n^2 , \tag{2}$$

$$\sigma_{n+1}^2=\mu^2(1-2M_n)^2\sigma_n^2+2\mu^2\sigma_n^4 , \tag{3}$$

where M_n is the mean value at iterate n and σ_n^2 is the corresponding dispersion. A further simplification can be realized by neglecting the σ_n^4 in Eq. (3). This we call the van Kampen expansion of the stochastic logistic map. For $\mu<3$, the fluctuations remain perturbations to the mean for all n and thus the above equations accurately represent the dynamics. However, there does exist a transient period (determined by where the mean equals half of the fixed point value) in which the fluctuations grow, thereby weakening the validity of the Gaussian closure approximation. This is an example of Kubo fluctuation enhancement [3].

Qualitatively, any closure scheme based on a Gaussian distribution for the chaotic logistic map is doomed to failure. We base this fact on a simple observation. The steady-state distribution also known as the invariant distribution has been solved analytically for this case [4] and is given by

$$P_{n\to\infty}=\frac{1}{\pi\sqrt{x(1-x)}} . \tag{4}$$

This is the well-known solution to the Perron-Frobenius equation [4] and it is obviously non-Gaussian.

The philosophy of this paper is that the Gaussian closure represents a two-variable parametrization of the evolving distribution function based on the initial distribution. Choosing instead a closure scheme based on a two-variable parametrization of the invariant distribution leads us to consider the following function:

$$P_n(x)=\frac{x^{p_n-1}(1-x)^{q_n-1}}{B(p_n,q_n)} . \tag{5}$$

This function, called the beta distribution, has several interesting properties. It is normalized over the [0,1] interval. The mean and dispersion of $P_n(x)$ are related to p and q by

$$M_n=\frac{p_n}{p_n+q_n} , \quad \sigma_n^2=\frac{p_n q_n}{(1+p_n+q_n)(p_n+q_n)^2} . \tag{6}$$

As p and q approach $\frac{1}{2}$, the steady-state distribution is realized. For $p,q \gg \frac{1}{2}$, the distribution looks Gaussian. In fact, as p and q get large, all cumulants greater than second order go to zero. Like the Gaussian distribution, $P_n(x)$ is parametrized by two variables. Hence we would not expect it to be an exact solution to the Perron-Frobenius equation and therefore Eq. (5) would not be expected to be accurate for all cumulants for all iterations. However, Eq. (5) should be a much better description of the dynamics for late iterations. In addition, because it looks Gaussian for large p and q, we would expect Eq. (5) to give an approximate description of the dynamics in the initial and transition regions.

Assuming the beta distribution accurately describes the stochastic evolution of the chaotic logistic map with Gaussian distributed initial conditions, the Nth-order moment at iterate n is simply

$$\langle x_n^N\rangle=\frac{B(N+p_n,q_n)}{B(p_n,q_n)} . \tag{7}$$

From Eq. (6) we have

$$p_n=\frac{M_n^2-M_n^3-M_n\sigma_n^2}{\sigma_n^2} , \quad q_n=\frac{(1-M_n)(M_n-M_n^2-\sigma_n^2)}{\sigma_n^2} . \tag{8}$$

From the recursive formula for the Γ function [i.e., $\Gamma(x+1)=x\Gamma(x)$] we obtain the following closure relation based on the beta distribution:

$$\langle x_n^N\rangle=\frac{(N-1+p_n)}{(N-1+p_n+q_n)}\langle x_n^{N-1}\rangle . \tag{9}$$

It is now straightforward to obtain a set of equations describing the evolution of the first- and second-order cumulants based on the beta closure scheme [Eq. (9)]. For simplicity, the coefficient in Eq. [9] is written as $C_N(M_n,\sigma_n^2)$. Hence

$$\langle x_n^N\rangle=C_N C_{N-1} C_{N-2} \cdots \langle x_n^2\rangle . \tag{10}$$

The evolution equations for the mean and the dispersion are given by

$$M_{n+1}=\mu M_n(1-M_n)-\mu\sigma_n^2 , \tag{11}$$

$$\begin{aligned}\sigma_{n+1}^2=\mu^2[1-2C_3(M_n,\sigma_n^2)\\ +C_4(M_n,\sigma_n^2)C_3(M_n,\sigma_n^2)](\sigma_n^2+M_n^2)\\ -[M_n(1-M_n)-\sigma_n^2]^2\mu^2 .\end{aligned} \tag{12}$$

It is important to realize that unlike the Gaussian closure approximation discussed earlier, higher-order cumulants within the beta closure approximation are not zero. Because of Eq. (10), all cumulants are defined in terms of the mean and dispersion and their dynamics is therefore known once the evolution of the mean and dispersion is known. Equations (11) and (12) are admittedly a more complicated set of equations than those encountered earlier [Eqs. (2) and (3)], but as we shall see, it gives a remarkably accurate description of the stochastic behavior of the logistic map.

The numerical procedure used to simulate the chaotic logistic map with stochastic initial conditions is carried out in a straightforward manner. We generate independent Gaussian distributed random deviates using the Box-Muller method. These deviates are characterized by their mean and standard deviation. The logistic map is iterated using these random numbers as initial conditions. Given that we are constrained not to exceed the edges of the unit interval, we always choose the mean and width of the Gaussian such that the mean is at least five standard deviations away from an endpoint. While in theory there is a nonzero probability of finding an initial point far from the mean, we never actually encounter this situation since the probability is vanishingly small. The map is iterated for 100 steps for each of 10^5 initial starting points. At every iterate n the averages of x_n^k are calculated for $k=1,\ldots,4$. We then compute the first- through fourth-order cumulants. The results obtained using a direct numerical simulation are then compared with those obtained by numerically solving the analytic Eqs. (11) and (12). All computations were performed in double precision on an SGI workstation.

We present here a comparison of the results of Eqs. (2) and (3) and Eqs. (11) and (12) with exact simulations for initial conditions given by $M_{n=0}=0.01$ and $\sigma_{n=0}^2=4\times10^{-6}$. Other sets of initial values for the mean and dispersion were run with values for the mean varying between 10^{-5} and 0.75 and values for the dispersion varying between 10^{-14} and 4×10^{-6}. The results we present here offer a representative sample.

Figures 1(a)–1(d) show the cumulants as a function of iterate for the initial parameters $M_{n=0}=0.01$ and $\sigma_{n=0}^2=4\times10^{-6}$ as calculated both by the beta distribution, the van Kampen expansion, and direct simulation.

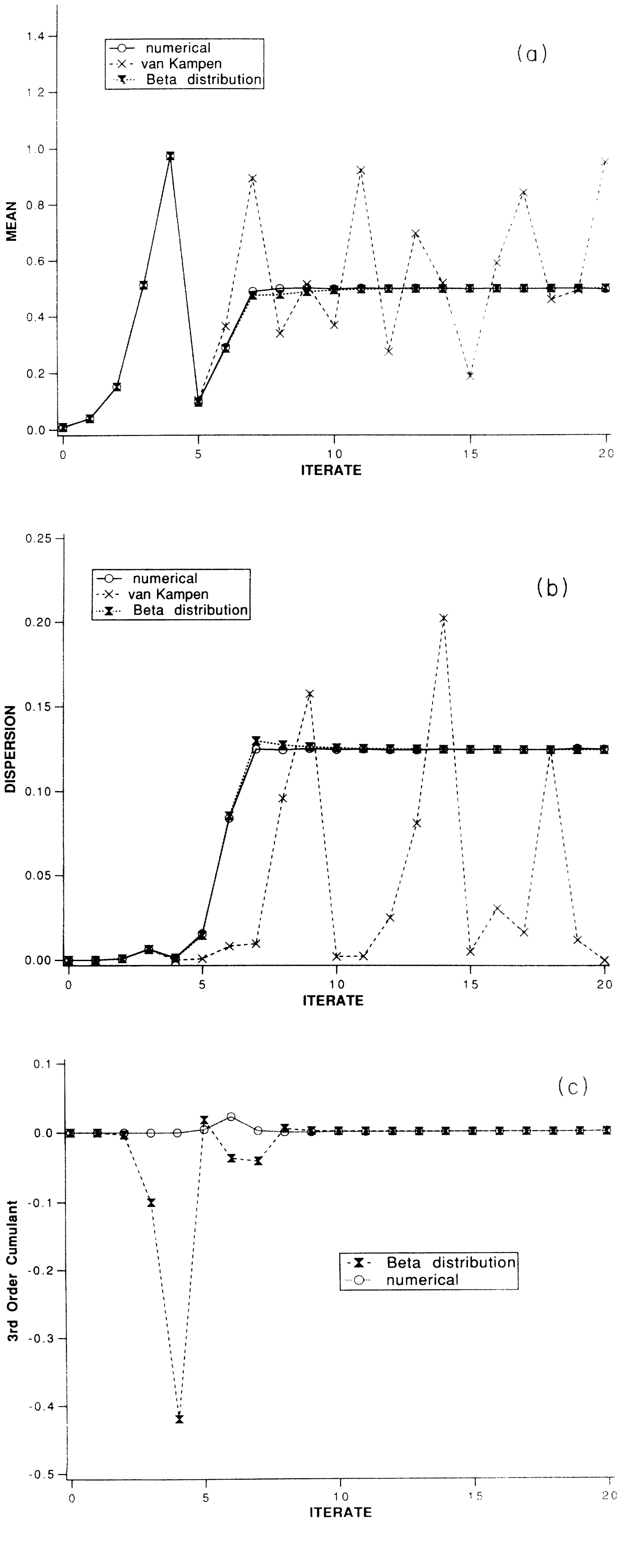

FIG. 1. $\mu=4$, $M_{n=0}=0.01$, $\sigma_{n=0}^2=4\times 10^{-6}$. (a) Mean vs iterate, (b) dispersion vs iterate, (c) third-order cumulant vs iterate, (d) fourth-order cumulant vs iterate. In (c) and (d) the circles represent the results of numerical simulation and crosses represent the results of the beta closure equations. The lines are a guide to the eye.

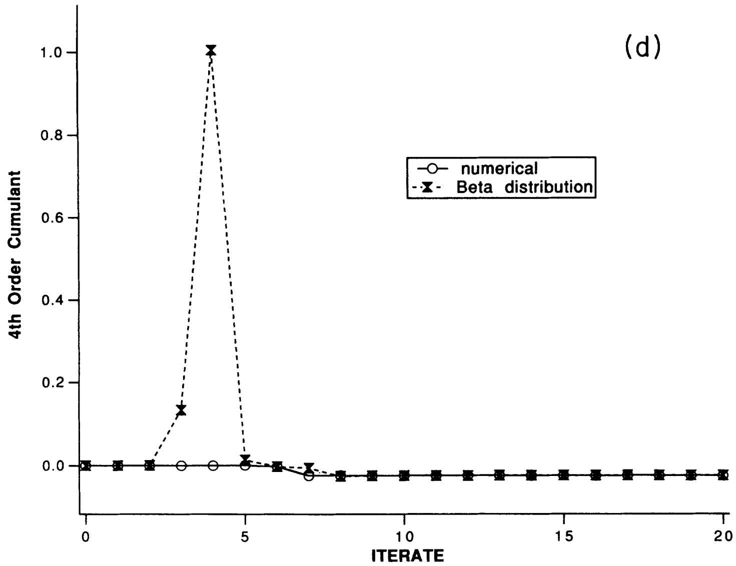

FIG. 1. (*Continued*).

The Gaussian closure equations are unstable for all the initial conditions we have looked at in the sense that eventually the iterate lies outside of the [0,1] interval. It is clear that the van Kampen expansion and direct simulation agree only initially. In particular, whereas direct simulations shows a definite steady state, the van Kampen expansion looks chaotic. We can trace this observation to the fact that the fluctuations no longer remain a small perturbation to the evolution of the mean, hence invalidating the linearization assumption used in deriving the van Kampen expansion. Finally, the beta closure approximation yields very accurate results for the mean and dispersion. In particular, the correct steady state is reached. It is evident from the direct simulations that there are three distinct regimes: an initial regime where the cumulants slowly grow or remain small, a transient regime where the cumulants increase rapidly, and a final regime (by about iterate 7) where the cumulants reach a steady-state value. In the case of the third-order cumulant it is zero while for the fourth-order cumulant it is -0.023. Both values are consistent with the invariant distribution. Like the Gaussian closure approximation for the logistic map with $\mu<3$, the beta closure is weakest (for higher-order cumulants) in the transient regime.

The evolution of the exact probability distribution beautifully illustrates the approach to steady state. The probability distribution obtained from this particular simulation clearly shows the initial narrow Gaussian distribution spreading as more of the phase space of the map is explored. Eventually, the distribution loses its Gaussian character as the iterates accumulate at the ends of the interval. Finally, the invariant distribution is attained as steady state has been reached. Since we have predictions for the mean and dispersion based on the beta closure approximation as a function of iterate, it is possible to compute p_n, q_n numerically and get a prediction for the beta probability distribution Eq. (5). A comparison can then be made between the probability distribution based on direct simulation versus beta closure. Figure 2 shows the result of that comparison. Note the excellent agreement (within statistical errors) for all iterates.

Suzuki [5] has analyzed theoretically the behavior of the relative diffusion (in our case the dispersion) for the chaotic logistic map using his scaling theory. We wish to compare our numerical results with his predictions. Scaling theory predicts that the dispersion should behave in the initial and transient regimes like

$$\sigma_n^2 \sim \frac{1-\cos(2^n\delta)}{2} . \tag{13}$$

In Fig. 3 we show comparisons between the numerical results for the dispersion versus the scaling theory prediction. The results show good qualitative agreement in the initial and transient regimes where the scaling theory applies. The agreement is completely wrong in the final steady-state regime. However, this is to be expected since the theory is not relevant in the steady state. Of particular interest is the fact that the iterate at which the transient regime occurs, and hence when the Gaussian and van Kampen expansions are breaking down, is predicted to be where the scaling function is order 1. That is,

$$n_c \sim \frac{-\ln(\delta)}{\ln(2)} . \tag{14}$$

An estimate based on Eq. (14) yields an iterate of 7 for an initial dispersion of 4×10^{-6} and 15 for an initial dispersion of 10^{-10}. Both agree with the numerical results.

The results of our calculations can now be summarized. The linear Gaussian closure gives the approximate early ($n \ll n_c$) evolution of the logistic map but breaks down after a certain number of iterations. The breakdown occurs when the fluctuations rapidly grow and become comparable to the mean. This is the transient regime identified by Suzuki scaling theory ($n \sim n_c$). The transient regime is the first indication of a complete breakdown of the Gaussian closure scheme. Finally, the stochastic system enters a steady-state regime characterized by the invariant distribution ($n \gg n_c$). Interestingly

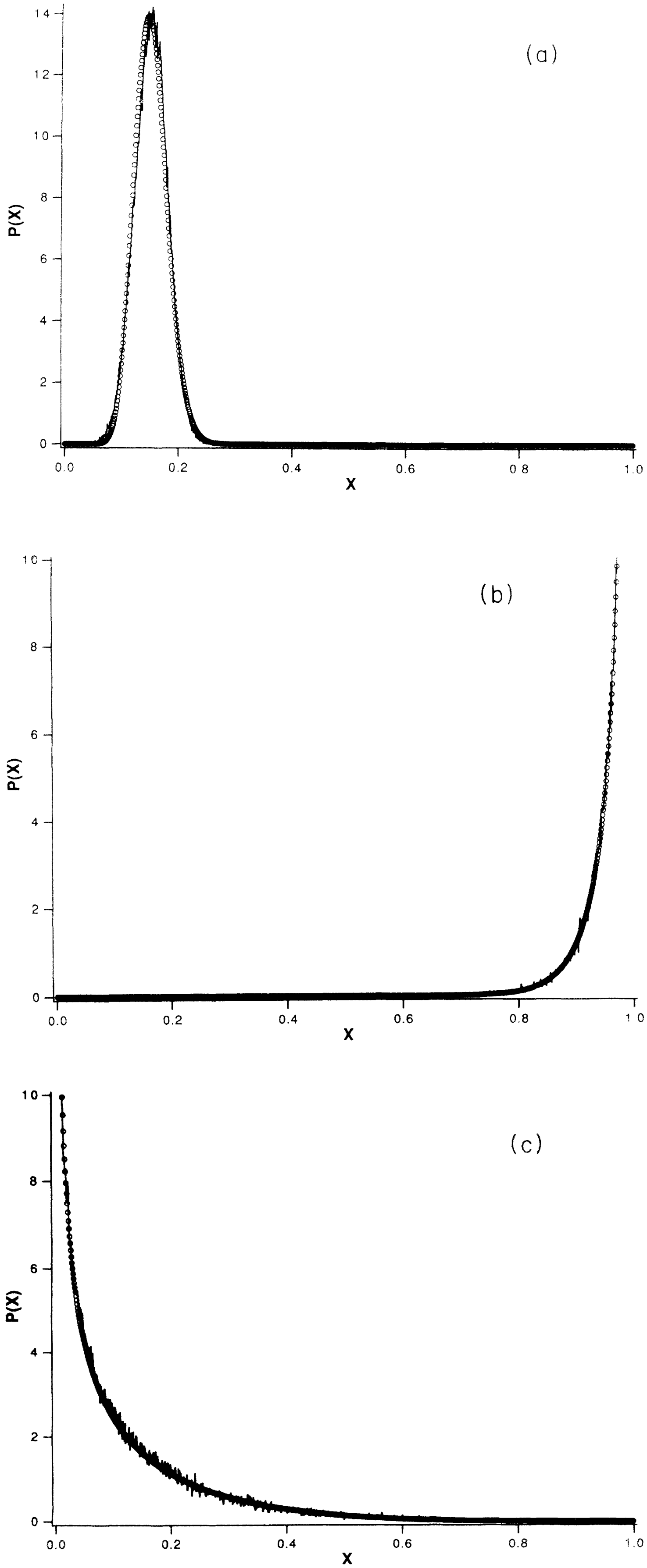

FIG. 2. Probability distribution for $\mu=4$, $M_{n=0}=10^{-2}$, $\sigma_{n=0}^2=4\times10^{-6}$: numerical simulations vs beta function for p and q obtained analytically. (a) Iterate 2 ($p=23.4794$, $q=131.1703$); (b) iterate 4 ($p=18.9786$, $q=0.5$); (c) iterate 5 ($p=0.4671$, $q=4.4421$); (d) iterate 15 ($p=0.5$, $q=0.5$). The solid line represents the numerical simulation while the circles represent the analytic result of Eq. (5).

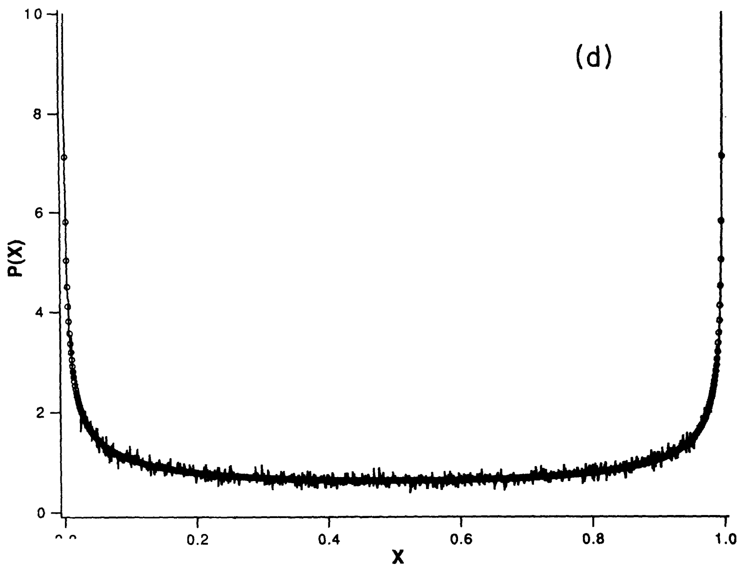

FIG. 2. (*Continued*).

enough, by basing a closure scheme on the steady-state distribution instead of the initial distribution, we have been able to come up with a closure scheme, albeit not perfect, that gives very accurate results for the mean and dispersion over all three regimes. In addition, higher-order cumulants are calculated properly in the initial and final regimes. However, any closure scheme based on a two-variable parameter set is not going to get everything right and the evidence for this is the differences between theory and experiment in the transient regime of the higher-order cumulants. It is important to realize that what we have introduced is a different type of closure scheme than is typically encountered. The beta closure scheme is nonperturbative in the sense that it assumes nothing about the relative magnitudes of higher- to lower-order moments. Instead, it only assumes that they can be related to the mean and dispersion in a straightforward way.

As far as the dynamic evolution of a clump of Gaussian distributed particles is concerned, the above results can be easily interpreted. The three regimes are characterized by the clump behaving initially like a coherent object whose dispersion or width remains small while its mean evolves according to the logistic map without fluctuations. The second regime is characterized by the particles in the clump diverging exponentially from each other. This is the transient regime described by the Suzuki scaling theory. Finally, the clump distribution set-

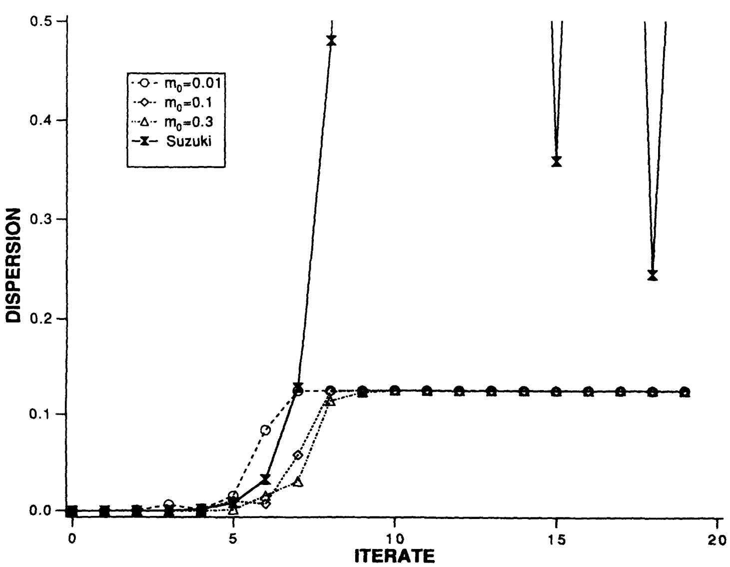

FIG. 3. Dispersion vs iterate for various means and an initial dispersion of 4×10^{-6}. The result of the Suzuki scaling formula is shown for comparison.

tles to a final state defined by its filling the [0,1] interval with accumulations near the endpoints.

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